ξ = viscosity parameter, $T_c^{1/6}/M^{1/2} P_c^{2/3}$
π = pressure, atm.
σ = collision diameter, Å
φ = azimuthal angle between the axes of the two dipoles
$\varphi(r)$ = Stockmayer potential, Equation (6)
$\Omega^{(1,1)*}[T_N]$ = reduced collision integral for the Lennard-Jones potential
$\Omega^{(2,2)*}[T_N]$ = reduced collision integral for the Lennard-Jones potential
$\Omega^{(1,1)*}[T_N, \delta^*]$ = reduced collision integral for the Stockmayer potential
$\Omega^{(2,2)*}[T_N, \delta^*]$ = reduced collision integral for the Stockmayer potential

LITERATURE CITED

1. Chen, N. H., and D. F. Othmer, *J. Chem. Eng. Data*, **7**, 37 (1962).
2. Flynn, L. W., and George Thodos, *A.I.Ch.E. Journal*, **8**, 362 (1962).
3. Hirschfelder, J. O., C. F. Curtiss, and R. B. Bird, "Molecular Theory of Gases and Liquids," Wiley, New York (1954).
4. Lydersen, A. L., R. A. Greenkorn, and O. A. Hougen, *Wisconsin Univ. Eng. Exp. Sta. Rept. No. 4* (October, 1955).
5. Mason, E. A., and L. Monchick, *J. Chem. Phys.*, **36**, 1622 (1962).
6. *Ibid.*, p. 2746.
7. *Ibid.*, **35**, 1676 (1961).
8. Pochettino, A., *Nuevo cimento*, **8**, 5, (1914).
9. Rowlinson, J. S., *Trans. Faraday Soc.*, **45**, 974 (1949).
10. Stiel, L. I., and George Thodos, *A.I.Ch.E. Journal*, **8**, 229 (1962).
11. ———, *J. Chem. Eng. Data*, **7**, 234 (1962).

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A Numerical Method for the Solution of the Energy Equation for Steady Turbulent Heat Transfer

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A method is presented for solving the basic energy expression for the temperature distribution for turbulent heat transfer in a circular conduit with a digital computer. The method assumes a knowledge of the velocity distribution, the eddy conductivity term as a function of the radial position, and a constant wall temperature. For illustrative purposes only a dimensionless expression is derived from Reichardt's relationship which gives the eddy conductivity term as a function of radial position with Reynolds and Prandtl numbers as parameters.

Incorporated in the computer program is Richardson's three-point extrapolation formula which permits the determination of a more accurate eigenvalue from three previously computed values which were obtained by using three different values of the reduced radial increment. Agreement with the Crank-Nicholson method is quite good, although the method with eigenvalues is considered to be superior. Eigenvalues determined from the analytical solution of the energy expression for the laminar flow case and those computed by the numerical method are in excellent agreement. This is indicative of the accuracy and reliability of the proposed numerical method.

In heat transfer studies involving the transfer of heat to or from fluids flowing in circular conduits it has always been of interest to be able to solve the appropriate energy expression for the temperature distribution or profile in both laminar and turbulent flow regions. The energy equation in cylindrical coordinates can be written for the turbulent flow condition as

$$\rho\, C_p\, u \frac{\partial T}{\partial y} = \frac{1}{r} \frac{\partial}{\partial r}\left[r(k + k_E) \frac{\partial T}{\partial r} \right] \qquad (1)$$

or as

$$u \frac{\partial T}{\partial y} = \frac{1}{r} \frac{\partial}{\partial r}\left[r\,(\alpha + \alpha_E) \frac{\partial T}{\partial r} \right] \qquad (1a)$$

where

$$\alpha = \frac{k}{\rho\, C_p}$$

$$\alpha_E = \frac{k_E}{\rho\, C_p}$$

Equation (1a) can be expressed in dimensionless form as

$$U \frac{\partial \theta}{\partial Z} = \frac{2}{X} \frac{\partial}{\partial X}\left[X\, \epsilon \frac{\partial \theta}{\partial X} \right] \qquad (2)$$

where the following dimensionless groups have been used:

$$\theta = \frac{T_w - T}{T_w - T_o}; \quad U = \frac{u}{\bar{u}}; \quad Z = \frac{L\alpha}{\bar{u}\, R^2} = \frac{2L}{N_{Re}\, N_{Pr}\, D}$$

$$X = \frac{r}{R}; \quad \epsilon = \frac{\alpha + \alpha_E}{\alpha} = 1 + \frac{\alpha_E}{\alpha}$$

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The well-known Graetz problem, as presented in detail by Jakob (*1*), was an analytical solution of Equation (2) for the laminar flow case. Under these conditions (U) becomes a parabolic function of (X) and $\epsilon = 1$. Owing to the extremely laborious nature of the computation, only the first three eigenvalues and eigenfunctions were given. These eigenvalues and eigenfunctions were recently revised and extended to order fifteen by Brown (*2*) and Larkin (3) with the aid of a digital computer. The Graetz problem has been extended by Siegel, Sparrow, and Hallman (*4*) to include steady laminar flow with any prescribed wall heat flux.

The analytical solution of the equation when applied to turbulent flow is impossible to obtain because of the nonlinearity of the velocity and eddy terms. Schlinger (*5*) and Sleicher (*6*) obtained solutions to Equation (2) for turbulent flow conditions using analogue computers. The applications of their solutions are confined to the specific eddy diffusivity functions and velocity profiles used in their analyses.

In the present work a numerical procedure has been formulated to solve Equation (2) for the turbulent flow case with a digital computer. A program can be written to obtain the temperature profile from any given complex eddy diffusivity function and velocity distribution if so desired. The heat transfer to a fully developed turbulent flow of air in a pipe at constant wall temperature is used in this work. The method presented could be extended to include the case of arbitrary wall temperature and heat flux from the method developed by Tribus and Klein (*7*). This procedure can of course be utilized for other problems in transport phenomena of an analogous nature such as the calculation of concentration profiles when the eddy diffusivity function or data are known.

VELOCITY PROFILE AND EDDY DIFFUSIVITY FUNCTIONS

The dimensionless velocity (U) and the eddy term (ϵ) in Equation (2) are functions of the dimensionless position (X) which must be known or determined before the equation can be solved for the temperature distribution. Experimental data for the velocity distribution of air flowing in circular ducts have been published by Nikuradse (*8*), Laufer (*9*), and Deissler (*10*). A mathematical expression for the velocity distribution has been presented by Hobler (*11*). This velocity profile expression in the turbulent region has been checked by the authors at various Reynolds numbers by comparing the calculated profiles with actual velocity profile data taken in heat transfer equipment and in a wetted wall column. The agreement is very good in all cases studied. This expression has the form

$$\frac{u}{\bar{u}} = \left[\frac{2(1-X)}{1-\psi\,(X)^s} - (1-X)^2 \right] \frac{u_{\max}}{\bar{u}} \qquad (3)$$

where

$$\psi = \left[1 - \frac{32}{(4f)N_{Re}} \right] \frac{u_{\max}}{\bar{u}} \qquad (4)$$

$$s = 2.7 \left[\frac{(1-X) + 0.2\,(\psi - 0.9)}{(1-X) + 0.02} \right] \qquad (5)$$

The exact function of the eddy term (ϵ) in Equation (2) is still in the stage of uncertainty. Most of the empirical expressions were developed from Prandtl's mixing length theory and the analogy between heat and momentum transfer as presented by Karman (*12*), Jenkins (*13*), Deissler (*14*), and Reichardt (*15*). Sleicher (*16*) presented values of α_E/ν vs. (X), calculated from experimental results, for different values of Reynolds numbers.

In order to illustrate the method, and in view of the lack of eddy thermal diffusivity data as a function of radial position, the dimensionless eddy term in Equation (2) can be derived from Reichardt's expression for the eddy viscosity which is

$$\frac{\nu_E}{(\nu) N_{Re^*}} = \frac{K}{3} \left[(0.5 + X^2)\ (1 - X^2) \right] \qquad (6)$$

The symbol (K) represents a universal constant known from the mixing length theory. Its value has been found to vary from 0.36 to 0.40. Starting with Reichardt's expression, the following relationships have been utilized:

$$N_{Re^*} = \frac{(R)\ (u^*)}{\nu} \qquad (7)$$

$$u^* = \sqrt{\frac{\tau w}{\rho}} \qquad (8)$$

$$\tau w = -\frac{\Delta P\ R}{2\ L} \qquad (9)$$

$$\frac{-\Delta P}{L} = \frac{f\ \bar{u}^2\ \rho}{R} \qquad (10)$$

$$f = \frac{0.184}{4(N_{Re})^{0.2}} \qquad (11)$$

$$\gamma = \frac{\alpha_E}{\nu_E} \qquad (12)$$

By the use of expressions (7) through (12) Equation (6) can be transformed to give

$$\frac{\alpha_E}{\alpha} = 0.005 \left[(N_{Pr})\,(\gamma)\,(N_{Re})^{0.9}\ (1 + 2\,X^2)\,(1 - X^2) \right] \qquad (13)$$

Since $\epsilon = 1 + \frac{\alpha_E}{\alpha}$, Equation (13) in final form becomes

$$\epsilon = 0.005[(N_{Pr})\,(\gamma)\,(N_{Re})^{0.9}\ (1 + 2\,X^2)\,(1 - X^2)] + 1 \qquad (14)$$

This shows that the dimensionless eddy term in Equation (2) is a polynomial function of radial position with the Prandtl number, Reynolds number, and (γ) as parameters. (γ) has been also found to be a function of position and fluid characteristics, and its evaluation has been discussed by Jenkins (*13*). Some experimental results have been recently given by Sleicher (*16*). (γ) in Equation (14) has been taken to be a constant and equal to 1.0 over the entire cross-sectional area in this instance owing to the scarcity of data concerning the variation of (γ) with radial position. Actual values of (γ) as a function of radial position are currently being investigated.

PROCEDURE FOR NUMERICAL SOLUTION

For fully developed turbulent flow in a straight pipe Equation (2) is expanded to give

$$U(X)\,\frac{\partial\theta}{\partial Z} = 2\left[\epsilon'(X) + \frac{\epsilon(X)}{X} \right] \frac{\partial\theta}{\partial X} + 2\epsilon(X)\,\frac{\partial^2\theta}{\partial X^2} \qquad (15)$$

with boundary conditions

$$\theta(X,Z) = 1, \quad Z \leq 0$$
$$\theta(1,Z) = 0, \quad Z > 0$$
$$\theta'(0,Z) = 0, \quad Z \geq 0$$

The solution, with the method of separation of variables, yields an expression containing eigenvalues and eigenfunctions as follows:

$$\theta(X,Z) = \sum_{0}^{\infty} C_n R_n(X)\, e^{-\lambda_n^2 Z} \qquad (16)$$

in which

$$R''_n(X) + \left[\frac{1}{X} + \frac{\epsilon'(X)}{\epsilon(X)}\right] R'_n(X) + \frac{\lambda_n^2 U(X)}{2\epsilon(X)} R_n(X) = 0 \quad (17)$$

with boundary conditions

$$R_n(0) = 1$$
$$R_n(1) = 0$$
$$R'_n(0) = 0$$

Equation (17) can be rearranged to give

$$\frac{d}{dX}[X\,\epsilon(X)\,R_n'(X)] + \frac{1}{2}\lambda_n^2 U(X)\,X\,R_n(X) = 0 \quad (18)$$

This is recognized as a differential equation of the Sturm-Liouville system. The eigenfunctions $R_n(X)$ have been proved to be orthogonal. At $Z = 0$ Equation (16) becomes

$$\theta(X,0) = 1 = \sum_0^\infty C_n R_n(X) \quad (19)$$

The value of C_n is obtained by using the orthogonal properties of the functions $R_n(X)$. Thus

$$C_n = \frac{\int_0^1 U(X)\,X\,R_n(X)\,dX}{\int_0^1 U(X)\,X\,R_n^2(X)\,dX} \quad (20)$$

$R_n(X)$ and λ_n^2 are solved numerically by trial and error and C_n determined by numerical integration. The temperature distribution can then be obtained from Equation (16).

It is noted that an indeterminacy in Equation (17) is present at $X = 0$. This indeterminacy is eliminated by L'Hospital's rule (*17*). For $X = 0$ Equation (17) is reduced to

$$R_n''(0) + \frac{\lambda_n^2 U(0)}{4\,\epsilon(0)} = 0 \quad (21)$$

The derivatives in this differential equation are approximated by Lagrange's three-point interpolation formula. At the center, where $X = 0$, Equation (21) becomes

$$\frac{R_{n,-1} - 2R_{n,0} + R_{n,1}}{h^2} + \left[\frac{\lambda_n^2 U(0)}{4\,\epsilon(0)}\right] R_{n,0} = 0 \quad (22)$$

Because of symmetry at the center

$$R_{n,-1} = R_{n,1}$$

and from the fourth boundary condition

$$R_{n,0} = 1$$

Equation (22) is rearranged to compute $R_{n,1}$ as indicated:

$$R_{n,1} = 1 - \frac{h^2 \lambda_n^2 U(0)}{8\,\epsilon(0)} \quad (23)$$

By the same means Equation (17) can be transformed for the computation of all other points of $R_n(X)$. Thus

$$R_{n,i+1} = \frac{\left[2 - \frac{\lambda_n^2 h^2 U(X_i)}{2\,\epsilon(X_i)}\right] R_{n,i} + \left[\frac{h\,\epsilon'(X_i)}{2\,\epsilon(X_i)} + \frac{1}{2i} - 1\right] R_{n,i-1}}{\left[1 + \frac{1}{2i} + \frac{h\,\epsilon'(X_i)}{2\,\epsilon(X_i)}\right]} \quad (24)$$

where $i = 1, 2, 3, \ldots m$

$$0 < (X_i = ih) < 1$$

If a value for λ_n^2 is assumed, $R_{n,1}$ can be determined immediately from Equation (23) for a given value of h. $R_{n,i+1}$ is then obtained by iteration, with $R_{n,0} = 1$ and the computed $R_{n,1}$ used as starting values. Several trials for λ_n^2 are necessary in order to obtain a value of λ_n^2 that corresponds to a value of $R_n(1) = 0$. A second subscript with parentheses is used to identify the order of the trial, (i.e., $\lambda^2_{n(1)}$, $\lambda^2_{n(2)}$, . . . $\lambda^2_{n,(j)}$). The true value of λ_n^2 should generate a $R_{n,m}$ or $R_n(1)$ which is the other boundary value and should be equal to zero in accordance with the fifth boundary condition. In numerical solutions a residue in $R_{n,m}$ is inevitable. However the allowable residue can be designated in accordance with the desired accuracy. The method of false position (*18*) was found to be very applicable for the rapid convergence of λ_n^2 towards the final value that corresponds to $R_n(1) = 0$.

The values of λ_n^2 thus obtained may have a significant difference depending upon the values chosen for (h). A small value of (h) may minimize the truncation error but increase the round-off error. The interaction between these two sources of error results in an optimum value of (h) which corresponds to the best possible value for λ_n^2. For this purpose Richardson's three-point extrapolation formula (*19*) is applicable for finding the best value of λ_n^2 from the three values of λ_n^2 which were obtained by using three different values of (h). This expression has the following form:

$$\lambda_n^2 = \left[\frac{(h_2^2)(h_3^2)}{(h_1^2 - h_2^2)(h_1^2 - h_3^2)}\right]\lambda^2_{n,h_1} + \left[\frac{(h_1^2)(h_3^2)}{(h_2^2 - h_1^2)(h_2^2 - h_3^2)}\right]\lambda^2_{n,h_2} + \left[\frac{(h_1^2)(h_2^2)}{(h_3^2 - h_1^2)(h_3^2 - h_2^2)}\right]\lambda^2_{n,h_3} \quad (25)$$

Once the optimum value of λ_n^2 has been determined, it can be used to compute the eigenfunctions by repeating the same procedure as used in the previous trials. The value of (h) should be readjusted in order to minimize

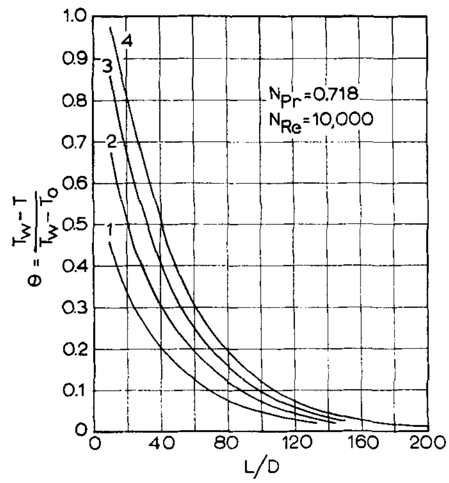

Fig. 1. Dimensionless temperature vs. L/D ratio for air at various radial positions (X). Curves 1, 2, 3, and 4 are for $X = 0.9$, 0.75, 0.50, and 0.0, respectively. $N_{Re} = 10{,}000$.

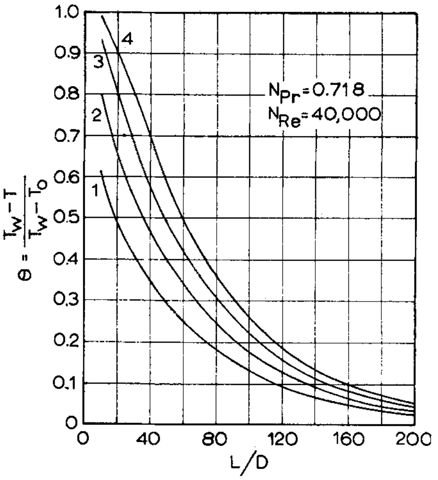

Fig. 2. Dimensionless temperature vs. L/D ratio for air at various radial positions (X). Curves 1, 2, 3, and 4 are for $X = 0.9, 0.75, 0.50$, and 0.0, respectively. $N_{Re} = 40{,}000$.

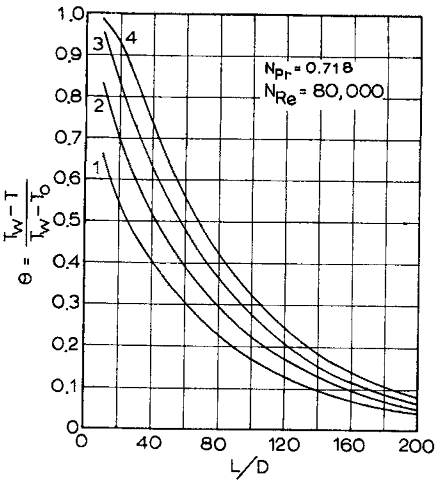

Fig. 3. Dimensionless temperature vs. L/D ratio for air at various radial positions (X). Curves 1, 2, 3, and 4 are for $X = 0.90, 0.75, 0.50$, and 0.0, respectively. $N_{Re} = 80{,}000$.

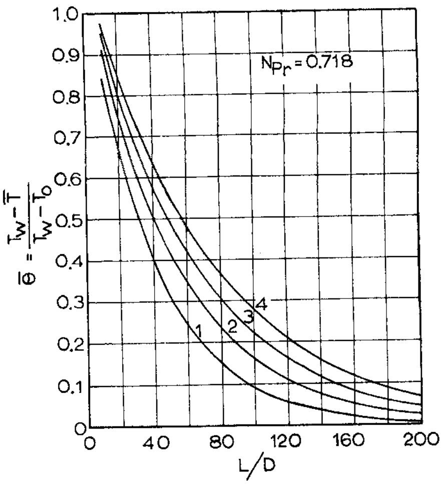

Fig. 4. Average dimensionless temperature vs. L/D ratio for air for various Reynolds numbers. Curves 1, 2, 3, and 4 are Reynolds numbers of 10,000, 20,000, 40,000, and 80,000, respectively.

the residue in $R_{n,m}$. The desired value of (h) may be obtained from the trial values of (h) and λ_n^2 by extrapolation. (C_n) is determined from the numerical integration of Equation (18) by Simpson's rule and the temperature distribution from Equation (16). These steps can all be incorporated into one program after λ_n^2 has been determined.

DISCUSSION AND RESULTS

The eigenvalues and eigenfunctions for the case of laminar flow in a circular conduit were computed with this numerical method. A comparison of the first six eigenvalues computed from the numerical solution with those obtained by Brown (2) using Graetz's analytical solution was found to be in excellent agreement. A deviation of only 0.24% in the sixth eigenvalue was observed. The eigenfunctions determined from the numerical solution deviate slightly in the fourth decimal place when compared with Brown's data using a value of $(h) = 0.01$.

Since the basic energy expression, Equation (2), is a parabolic partial differential equation, it can also be solved implicitly by the finite difference method for the temperature distribution if the functions of the eddy thermal diffusivity and the velocity profile are known. The Crank-Nicholson method (20) was used for comparative purposes. The temperature distribution, determined by these two numerical methods, were in fairly good agreement. In order to obtain a higher degree of accuracy with the Crank-Nicholson method very small mesh sizes in both the radial and axial directions are required. This results in an excessive amount of computation time. The method presented is considered to be superior to the Crank-Nicholson method owing to the elimination of the approximations in the axial direction.

The first three eigenvalues and eigenfunctions and the constants (C_n) corresponding to Reynolds numbers of 10,000, 40,000, and 80,000 respectively were computed and are presented in Table 1. The programs were written in Fortran language and computed on an IBM-1620. The trial values of (h) were chosen to be 0.05, 0.02, and 0.01. The value of (h) for computing eigenfunctions and the constant (C_n) was 0.005. For a given Reynolds number in Table 1 the computing time for each final eigenvalue was approximately 19 min. or 57 min. for the three eigenvalues shown. The three eigenfunctions, constants, and the temperature distribution required a total computation time of about 23 min.

The numerical method is further illustrated in Figures 1 through 5 for air with a Prandtl number of 0.718. Figures 1, 2, and 3 present the dimensionless temperature distribution (θ) plotted against the dimensionless (L/D) term for various values of Reynolds numbers and radial positions. Actually (θ) is plotted against the dimensionless (Z) term of Equation (2). However since the Prandtl and Reynolds numbers are constant in each figure, the (L/D) term is used. Figure 4 presents an average dimensionless temperature vs. (L/D) for air at different Reynolds numbers. The average temperature was obtained by numerical integration over the cross-sectional area utilizing the temperature and velocity profiles. Figure 5 is a plot of the asymptotic Nusselt number vs. Reynolds number in turbulent flow for air. The asymptotic Nusselt number has been calculated from the first eigenvalue computed by the numerical method with the fol-

TABLE 1. EIGENVALUES AND EIGENFUNCTIONS FOR VARIOUS REYNOLDS NUMBERS

$N_{Re} = 10{,}000$

$\lambda_1^2 = 88.008$, $C_1 = 1.3584$	$\lambda_2^2 = 646.14$, $C_2 = -0.56892$	$\lambda_3^2 = 1730.86$, $C_3 = 0.37790$	
X	$R_1(X)$	$R_2(X)$	$R_3(X)$
0	1.0	1.0	1.0
0.1	0.99107	0.93532	0.83145
0.2	0.96512	0.75769	0.41859
0.3	0.92419	0.50806	−0.02978
0.4	0.87064	0.23520	−0.32399
0.5	0.80630	−0.01831	−0.38171
0.6	0.73172	−0.22228	−0.23345
0.7	0.64510	−0.35803	0.02084
0.8	0.53983	−0.41230	0.25819
0.9	0.39312	−0.36186	0.35145
1.0	0	0	0

$N_{Re} = 40{,}000$

$\lambda_1^2 = 230.81$, $C_1 = 1.3039$	$\lambda_2^2 = 1945.74$, $C_2 = -0.48929$	$\lambda_3^2 = 5261.76$, $C_3 = 0.32305$	
X	$R_1(X)$	$R_2(X)$	$R_3(X)$
0	1.0	1.0	1.0
0.1	0.99305	0.94211	0.84730
0.2	0.97283	0.78212	0.46669
0.3	0.94085	0.55419	0.03521
0.4	0.89880	0.29912	−0.27855
0.5	0.84791	0.05297	−0.38754
0.6	0.78824	−0.15792	−0.29626
0.7	0.71780	−0.31611	−0.07204
0.8	0.63005	−0.40775	0.18755
0.9	0.50228	−0.40729	0.36297
1.0	0	0	0

$N_{Re} = 80{,}000$

$\lambda_1^2 = 398.44$, $C_1 = 1.2874$	$\lambda_2^2 = 3517.44$, $C_2 = -0.46244$	$\lambda_3^2 = 9535.68$, $C_3 = 0.30536$	
X	$R_1(X)$	$R_2(X)$	$R_3(X)$
0	1.0	1.0	1.0
0.1	0.99359	0.94407	0.85198
0.2	0.97494	0.78919	0.48113
0.3	0.94542	0.56767	0.05546
0.4	0.90655	0.31811	−0.26281
0.5	0.85941	0.07475	−0.38621
0.6	0.80397	−0.13726	−0.31275
0.7	0.73826	−0.30112	−0.10061
0.8	0.65592	−0.40334	0.16139
0.9	0.53481	−0.41911	0.36055
1.0	0	0	0

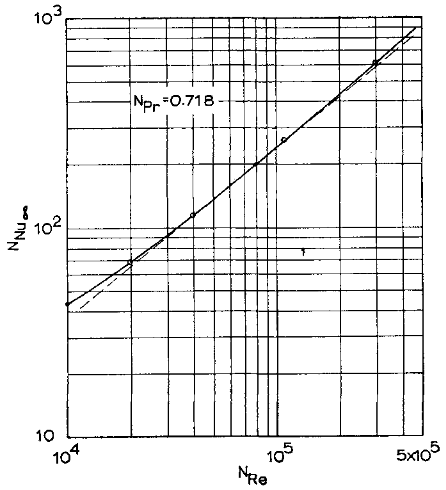

Fig. 5. Asymptotic Nusselt number vs. Reynolds number for air.

lowing relationship for well-developed turbulent flow at constant wall temperature:

$$N_{Nu_\infty} = 1/2\,[\lambda_1^2] \tag{26}$$

The slight curvature noted tends to confirm the fact that the Nusselt number is not a simple power function of the Reynolds and Prandtl number except over limited ranges. Over the range of Reynolds numbers plotted the curve can be approximated by a straight line having a slope of 0.80.

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NOTATION

C_p = heat capacity
C_n = constant, Equation (2)
D = diameter of conduit, $(2R)$
f = Fanning friction factor
h = increment in (X) direction $= 1/m$
i = increment designation in (X) direction
j = order of trials in final eigenvalue determination
K = constant, Equation (6)
k = thermal conductivity
k_E = eddy thermal conductivity
L = length of test section
m = total intervals along (X), Equation (24)
N_{Nu_∞} = asymptotic Nusselt number
N_{Pr} = Prandtl number
N_{Re} = Reynolds number
N_{Re^*} = Reynolds number using the friction velocity (u^*)
n = order of eigenvalues or eigenfunctions
ΔP = pressure drop over length (L)
R = radius of conduit
r = radial distance from center of conduit
$R_{n,i}$ = point value of eigenfunction of order (n)
$R_n(X)$ = eigenfunction
$R_n'(X)$ = first derivative of eigenfunction
$R_n''(X)$ = second derivative of eigenfunction
s = exponent in the velocity profile expression, Equations (3) and (5)
T = temperature at any point
$\overline{T}$ = average temperature over cross-sectional area
T_o = reference or inlet temperature
T_w = temperature at the wall
U = dimensionless velocity, Equation (2)
u = axial velocity, a function of radial position
$\overline{u}$ = average velocity in conduit
u_{max} = maximum velocity (velocity at $X = 0$)
u^* = friction velocity, Equation (8)
X = dimensionless radial position, Equations (2) and (6)
y = axial distance
Z = dimensionless length, Equation (2)

Greek Letters

α = thermal diffusivity, Equation (1*a*)
α_E = eddy thermal diffusivity, Equation (1*a*)
γ = ratio of eddy thermal diffusivity to eddy viscosity
ϵ = dimensionless term including molecular and eddy thermal diffusivity, Equation (2)
$\epsilon'(X)$ = first derivative of dimensionless eddy term
θ = dimensionless temperature, Equation (2)
$\overline{\theta}$ = average dimensionless temperature
θ' = first derivative of dimensionless temperature term
λ_n^2 = eigenvalue
$\lambda_{n,h}^2$ = eigenvalue determined from a particular value of h
ν = kinematic viscosity
ν_E = eddy viscosity
ρ = fluid density
τ_w = shear stress at the wall
ψ = term in the velocity profile expression defined by Equation (4)

LITERATURE CITED

1. Jakob, Max, "Heat Transfer," Vol. 1, p. 451, Wiley, New York (1949).
2. Brown, G. M., *A.I.Ch.E. Journal*, **6**, 179 (1960).
3. Larkin, B. K., *ibid.*, **7**, 530 (1961).
4. Siegel, R., E. M. Sparrow, and T. M. Hallman, *Appl. Sci. Res.*, **A7**, 386 (1958).
5. Schlinger, W. G., V. J. Berry, J. L. Mason, and B. H. Sage, *Ind. Eng. Chem.*, **45**, 662 (1953).
6. Sleicher, C. A., and M. Tribus, *Trans. Am. Soc. Mech. Engrs.*, **79**, 789 (1957).
7. Tribus, M., and J. Klein, "Heat Transfer, a Symposium held at the Univ. of Mich. during the Summer of 1952," p. 211, Engineering Research Institute, Univ. of Mich., Ann Arbor, Michigan (1953).
8. Nikuradse, J., in Donald Ross' Report No. *Nord 7958-246*, Ord. Res. Lab., The Pennsylvania State College, University Park, Pennsylvania (1952).
9. Laufer, J., *Natl. Advisory Comm. Aeronaut. Tech. Note 2954*, (1953).
10. Deissler, R. G., *Natl. Advisory Comm. Aeronaut. Tech. Note 2138* (1950).
11. Hobler, T., *Chemia Stosowana (Poland)*, **1**, 1 (1957); abstracted in *Brit. Chem. Eng.*, **3**, 329 (1958).
12. Von Karman, Th., *Trans. Am. Soc. Mech. Engrs.*, **61**, 705 (1939).
13. Jenkins, R., "Heat Transfer and Fluid Mechanics Institute Preprints," p. 147, Stanford University Press, Stanford, California (1951).
14. Deissler, R. G., *Natl. Advisory Comm. Aeronaut. Tech. Note 3145* (1954) .
15. Reichardt, H., in Hartnett, J. P., "Recent Advances in Heat and Mass Transfer," p. 223, McGraw-Hill, New York (1961).
16. Sleicher, C. A., *Trans. Am. Soc. Mech. Engrs.*, **80**, 693 (1958).
17. Fox, L., "The Numerical Solution of Two-Point Boundary Value Problems," p. 100, The Clarendon Press, Oxford, England (1957).
18. Hildebrand, K. B., "Introduction to Numerical Analysis," p. 446, McGraw-Hill, New York (1956).
19. Kopal, Z., "Numerical Analysis," p. 250, Wiley, New York (1955).
20. Forsythe, G. E., and W. R. Wasow, "Finite Difference Methods for Partial Differential Equations," p. 120, Wiley, New York (1960).

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